Mathematics HL Diagnostic test – Non–Calculator paper (Paper 1)

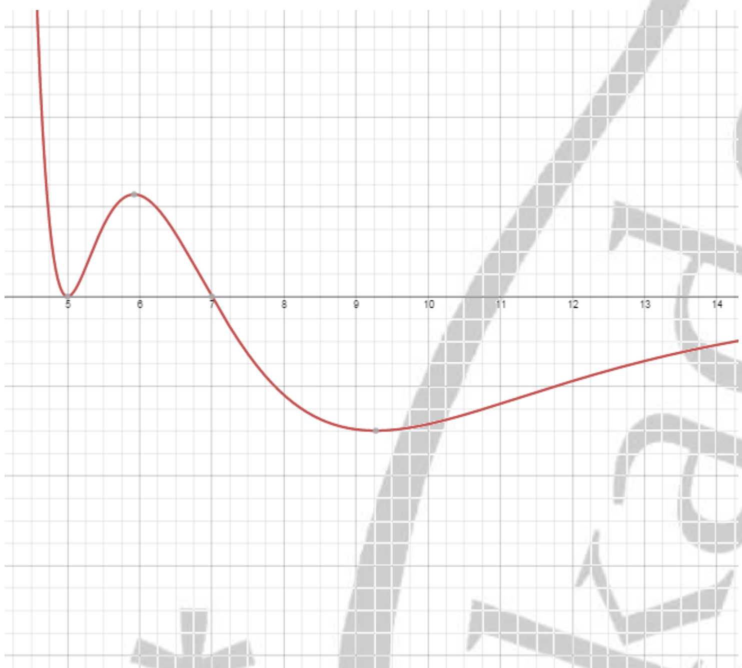
Section A

1. Consider the function $f(x) = 2\cos(x) - 1$ and $g(x) = x^2 - 7$.

a) Show that g(f(x)) is an even function. [2]

b) Find the range of g(f(x)). [3]

2. Given the function graph f(x) below,



a) Sketch $\frac{1}{f(x)}$. [3]

b) Sketch $|f(x)| - f(x)$. [3]

3. Solve the equation $\cos(90^\circ - m) = \sin(60^\circ - 2m)$ over $0^\circ < m < 360^\circ$. [6]

4. Given that the function $y = -|x - 0.5| + a$ is a probability distribution function for $0 \leq x \leq 1$,

a) Find the value of a. [4]

b) Find $P(0 \leq x \leq 0.25)$. [3]

5. In the expansion $(x + \sqrt{5})^n$, the coefficient of the x and x^3 terms are the same. Find n. [6]

6. Let $f(x) = x^2 - 3$ and $g(x) = -(x - 2)(x - 8)$.

a) Deduce the coordinates of points of intersection of f(x) and g(x) if they exist. [2]

b) Find the equations of the two common tangents between the two parabolas. [8]

7. The number of fish caught on a fishing trip is modeled by the Poisson distribution with mean λ. Let X be the number of fish caught during a particular trip. It's given that the probabilities P(X = 2), P(X = 3) and P(X = 4) form consecutive terms of an arithmetic sequence. Deduce the possible values of λ. [5]

8. A sequence is given by the recursive relation $T_1 = 1$, $T_{n+1} = 1 + \frac{6}{T_n}$ for all other positive–integer values of n. Prove by mathematical induction that $T_m < 3$ for all odd values of m. [6]

9. Consider the cubic polynomial $x^3 - 11x + 150$.

a) Show that x + 6 is a factor of the polynomial. [1]

b) Hence determine all 6 roots of the equation $z^6 - 11z^2 + 150 = 0$. [8]

Section B

10. Consider the graph $y = f(x) = \frac{5x-15}{x-4}$.

a) Sketch f(x), clearly indicating all asymptotes and intercepts.[3]

b) Let $g(x) = -(x - 3)(x - 10)$. Solve $f(x) \geq g(x)$. [5]

c) Consider the finite region between the x–axis, the y–axis and f(x). Find the area of the biggest rectangle that can be inscribed within the region, justifying that the value that you calculated is a maximum. [6]

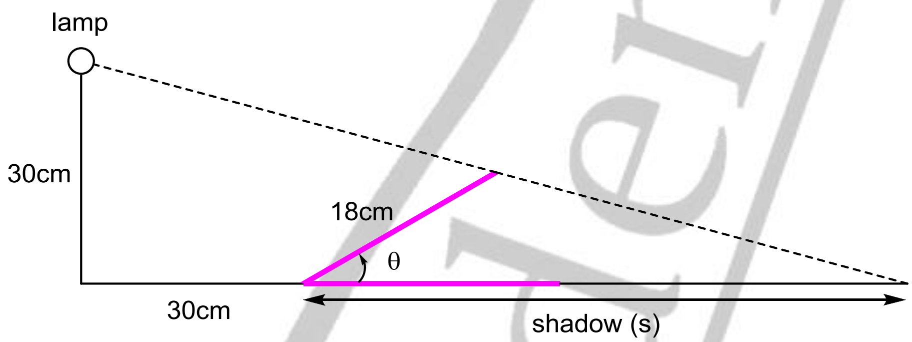
11. Consider the region confined by the x–axis of graph y = sin x between 0 to π.

a) Find the area of this region. [1]

b) Find the volume of revolution when this region is revolved 2π radians around the x–axis. [4]

c) Show that the volume of revolution when this region is revolved 2π radians around the y–axis is $2\pi^2$ cubic units. [10]

12. A book, placed 30 cm away from a 30 cm-tall desk lamp, is being opened:



a) By adding a suitable construction line, show that for $0 < \theta < \frac{\pi}{2}$, the length of the shadow, s, is given by

$s = \frac{90(sin\theta + cos\theta)}{5 - 3sin\theta}$ [4]

b) At t = 0 second, the book is closed. The book is now opened such that the angle θ increases at a constant rate of 1/3 radians per second.

i) Find the rate of decrease of the length of the shadow when $\theta = \frac{\pi}{2}$ radians. [5]

ii) Show that the angle θ that gives the longest possible shadow satisfies $\sin(2\theta) = \frac{m}{n}$ where m and n are integers to be found. [6]

13. Consider the two line equations

$$L_1: \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 7 \end{pmatrix} + s\begin{pmatrix} 3 \\ 5 \\ 4 \end{pmatrix} \text{ and } L_2: \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 11 \\ m \end{pmatrix} + t\begin{pmatrix} -2 \\ -1 \\ 2 \end{pmatrix}$$

a) Find the value of m such that L_1 and L_2 intersect, and find the point of intersection. [6]

You may use the value of m from part a) to attempt part b).

b) A plane P contains the origin and is parallel to both L_1 and L_2. Show that the equation of this plane is $2x - 2y + z = 0$. [2]

c) Find the range of values of m such that the plane in b) is closer to L_2 than to L_1. [8]

Mathematics HL Diagnostic test –Calculator paper (Paper 2)

Section A

1. 8 black balls and 4 green balls are placed in a bag, and 3 balls are drawn without replacement. Find the mean and standard deviation for the number of black balls within the 3–ball sample. [6]

2. Consider the three plane equations

P_1: $x + z = 4$
P_2: $4x + ky - z = 21$
P_3: $x - 6y + kz = 10$

There are two values of k that will lead to the system of equations not having a unique solution.

a) Find these two values of k. [4]

b) The system is consistent for only one of the values of k above. Find the general solution for this value of k. [4]

3. Determine the complex number z that satisfies $|z| = \sqrt{90}$ and $\arg(z + 6) = \frac{5\pi}{4}$. [5]

4. How many ways are there to seat 5 people within a row of 11 empty seats so that......

a) no one is immediately next to each other? [3]

b) 3 people are sitting together, while the other two are separated from each other and also from the group of 3? [3]

5. Given that $y = \cos^2 x(\sin x + k)$ where k is a constant, show that $\cos^2 x\frac{dy}{dx} = \cos^5 x - y\sin(2x)$. [5]

6. The curve C has equation $x^2 - y^2 = 8$. Determine the coordinates of the two points on C at which the normal passes through the point (0, 4). [6]

7. The body–mass index of American female adults is known to follow the normal distribution of mean 28 and standard deviation 6. A female is deemed obese if her body–mass index exceeds 30.

a) Find the chance that a group of 10 females contains 4 or more obese females. [4]

b) Given that a group of 10 females contain at least 4 obese females, find the probability that there are no more than 6 obese females. [3]

8. In a school cafeteria a group of students are surveyed about their preferences for peppering their soup (P) and creaming their coffee (C). Let P and C represent the proportion of students who pepper their soup and cream their coffee respectively. It's known that P(P) < P(C). Assume that P and C are independent events.

a) Find the value of P(P) and P(C) if $P(P \cap C) = P(P' \cap C') = \frac{91}{400}$. [4]

b) Given that a student likes to add condiment to only 1 of the two items, find the probability that it's pepper to soup. [3]

9. At 12:00 noon (t = 0) two boats start to travel according to the following vector equations (distances and times in km and hour, respectively)

B_1: r = (i + 7j) + t(4i + 2j) B_2: r = (5i + 4j) + t(5i + 3j)

a) Find the position vector of the point where they cross path? [5]

b) Find when they are closest to each other? [5]

Section B

10. (Calculator allowed) An object free–falls amidst air resistance, starting from rest, with the acceleration, as a function of time, given by $a(t) = 10e^{-0.25t}$, with all distances and time given in meters and seconds, respectively.

a) The terminal velocity of the object is velocity of the object at extremely large values of t. Find the terminal velocity of the object. [6]

b) Find the time when the object's velocity is 99% of the terminal velocity, giving your answer in the form of ln(k) where k is an integer. [3]

c) Find the average speed of the object over the 20 seconds [4].

d) How long does it take for the object to fall 1000 meters? [3]

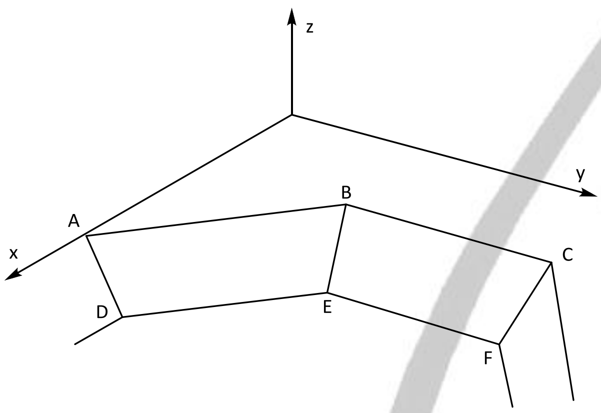
11. Consider the graph $y = \frac{x-b}{(x-a)(x-c)}$ where $0 < a < b < c$.

a) State the equations of the asymptotes. [3]

b) Use calculus to show that f(x) has no turning points. [8]

c) Sketch y = f(|x|+a), clearly indicating all graphical features in terms of a, b and c. [6]

12. (HL only, planes involved, calculator needed) Shown in the diagram is part of a swimming pool with slanted lateral sides. The points A, B, C and D have coordinates (14, 0, 0), (6, 8, 0), (6, 18, 0) and (14, 1, –6), respectively. The plane BCFE has a normal vector of 6i + k.



a) Find the equations of the plane BCFE and ABED. [5]

b) Find the obtuse angle between the two planes. [3]

c) Find a parametric equation for the line BE. [4]

d) Given the base DEF is horizontal,

i) find the coordinates of E. [2]

ii) find the area of trapezium ABDE. [5]

e) A spider wishes to crawl from point D to the midpoint of BC along the walls ABED and CBEF. Find the shortest distance that the spider has to crawl. [8]