FACTOR & REMAINDER THEOREM

First consider the following division where f(x) is a polynomial in x [ie it is the sum of terms in x where the powers of x are non-negative integers]:

quotient → $g(x)$

divisor → $ax+b \,\overline{)\, f(x)}$ ← dividend

$\vdots$

remainder → R

compare with:

$$\begin{array}{r} 69 \\ 5\overline{)349} \\ \underline{30} \\ 49 \\ \underline{45} \\ 4 \end{array}$$

$$\frac{349}{5} = 69\tfrac{4}{5}$$

$$\left(= 69 + \frac{4}{5}\right)$$

This gives: $$\frac{f(x)}{ax+b} = g(x) + \frac{R}{ax+b}$$

The remainder, R, is just a constant as it must have degree at least one less than the divisor. And since the divisor has degree 1, the remainder must have degree 0 - ie a constant.

If we now multiply through by (ax + b), we will get:

$$f(x) = g(x).(ax + b) + R$$

so if we replace x by the value that makes (ax + b) = 0, ie we replace x by $\frac{-b}{a}$, we will get:

$$f\left(-\frac{b}{a}\right) = g\left(-\frac{b}{a}\right) \cdot \underbrace{\left(a\left(-\frac{b}{a}\right) + b\right)}_{0} + R$$

$$\Rightarrow f\left(-\frac{b}{a}\right) = g\left(-\frac{b}{a}\right)(0) + R$$

$$\Rightarrow R = f\left(-\frac{b}{a}\right)$$

So if we divide a polynomial by a linear expression, the remainder is simply the value of the polynomial evaluated at the value of x that makes the linear expression equal 0.

This is the Remainder Theorem:

Remainder Theorem:

Given polynomial f(x), the remainder when f(x) is divided by (ax + b) is $f\left(\frac{-b}{a}\right)$.

Now if (ax + b) is actually a factor of f(x) then the remainder must be 0, so by the remainder theorem, $f\left(\frac{-b}{a}\right) = 0$.

We can actually use this idea to help us factorize a polynomial, by just 'randomly' substituting values into the polynomial until we get a 0. When we do, the linear expression that has that x-value as a root is a factor. This is the Factor Theorem:

Factor Theorem:

Given polynomial f(x), if $f\left(\frac{b}{a}\right) = 0$, then (ax - b) is a factor of f(x).

$x = \frac{b}{a} \Rightarrow ax = b \Rightarrow ax - b = 0$

The choice of x-values to substitute into the polynomial is not so random. You just need to try:

(1) + and - values of the factors of the constant term of f(x)

If none of these work,

(2) try fractions where the numerator are the values in (1) and the denominator are factors of the coefficient of the highest power term in f(x).

Examples

Factorise completely:

1) $x^3 - 6x^2 + 11x - 6$

Let $f(x) = x^3 - 6x^2 + 11x - 6$

$f(1) = 1 - 6 + 11 - 6 = 0$

$\Rightarrow (x-1)$ is a factor of $f(x)$ by the Factor Theorem.

$f(x) = (x-1)(x^2 - 5x + 6)$

$= (x-1)(x-2)(x-3)$ //

can find by algebraic division

2) $2x^3 - 3x^2 - 11x + 6$

Let $f(x) = 2x^3 - 3x^2 - 11x + 6$

$f(-2) = -16 - 12 + 22 + 6 = 0$

$\Rightarrow (x+2)$ is a factor of $f(x)$ by the Factor Theorem

$f(x) = (x+2)(2x^2 - 7x + 3)$

$= (x+2)(x-3)(2x-1)$ //

3) When divided by $x + 1$, the polynomial $ax^3 - x^2 - x + 6$ leaves a remainder of 4. Find the value of the constant a.

Let $f(x) = ax^3 - x^2 - x + 6$

Then $f(-1) = -a - 1 + 1 + 6 = 4$

$2 = a$

$a = 2$ //

4) When divided by $x - 1$, the polynomial $ax^3 + x^2 + bx - 4$ leaves a remainder of -6. Given that $x - 2$ is a factor of the polynomial, find the values of the constants a and b.

Let $f(x) = ax^3 + x^2 + bx - 4$

$f(1) = a + 1 + b - 4 = -6$

$$a + b = -3 \quad (1)$$

$$f(2) = 8a + 4 + 2b - 4 = 0$$

$$\Rightarrow \quad 8a + 2b = 0$$

$$\Rightarrow \quad 4a + b = 0 \quad (2)$$

$(2) - (1)$: $\quad 3a = 3$

$$a = 1$$

Into (1) : $\quad 1 + b = -3$

$$b = -4$$